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Note

A sufficient condition for a plane graph with maximum degree 6 to be class 1 [☆]

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ABSTRACT

A well-known conjecture of Vizing (the planar graph conjecture) states that every plane graph with maximum degree $\Delta \geq 6$ is edge Δ-colorable. Vizing himself showed that every plane graph with maximum degree $\Delta \geq 8$ is edge Δ-colorable. Zhang [L. Zhang, Every graph with maximum degree 7 is of class 1, Graphs Combin. 16 (2000) 467–495] and Sanders and Zhao [D. P. Sanders, Y. Zhao, Planar graphs of maximum degree seven are class 1, J. Combin. Theory Ser. B 83 (2001) 201–212] independently proved that every plane graph with maximum degree 7 is of class 1, i.e., edge 7-colorable. This note shows that every plane graph G with maximum degree 6 is edge 6-colorable if no vertex in G is incident with four faces of size 3.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. Call a graph G *planar* if it can be drawn on the plane so that its edges meet only at their ends. Any such drawing of a planar graph is called a *plane* graph. For a plane graph G, we use V, E, F and Δ to denote its vertex set, edge set, face set and maximum degree, respectively. For vertices $u, v, w \in V$, let $E_G(u)$ or $E(u)$ be the set of edges incident with u, $N(u)$ the set of vertices adjacent to u, and $N(u, v) = N(u) \cup N(v)$, $N(N(u)) = \{w | vw \in E, v \in N(u)\}$, $N(N(u, v)) = N(N(u)) \cup N(N(v))$. The *degree* of v in G, denoted $d_G(v)$ or $d(v)$, is the cardinality of $E(v)$. Call v a *k-vertex*, or a k^+*-vertex*, or a k^-*-vertex* if $d(v) = k$, or $d(v) \geq k$, or $d(v) \leq k$, respectively. For a face $f \in F$, the number of edges of the boundary of f (a cut-edge is counted twice), denoted $d(f)$, is called the *size* or *degree* of f. Call f a *k-face*, or a k^+*-face*, or a k^-*-face*, if $d(f) = k$, or $d(f) \geq k$, or $d(f) \leq k$, respectively. We write $f = [v_1 v_2 \cdots v_k]$ if $v_1, v_2, \ldots, v_k$ are consecutive vertices on f in a cyclic order, and say that f is a $(d(v_1), d(v_2), \ldots, d(v_k))$-*face*. An edge xy is called a $(d(x), d(y))$-edge, and x is called a $d(x)$-*neighbor* of y. Let $d_k(x)$ denote the number of k-neighbors of x, $F_k(x)$ the set of k-faces incident with x, and $f_k(x) = |F_k(x)|$.

An *edge k-coloring* of a graph $G = (V, E)$ is a mapping ϕ from E to the set of colors $\{1, 2, \ldots, k\}$ such that $\phi(e) \neq \phi(e')$ whenever e and e' are adjacent. The *chromatic index* $\chi'(G)$ is the smallest integer k such that G admits an edge k-coloring. The well-known Vizing theorem [3] on edge coloring asserts that, for every simple graph G, $\Delta \leq \chi'(G) \leq \Delta + 1$. G is said to be *class* 1 if $\chi'(G) = \Delta$ and *class* 2 if $\chi'(G) = \Delta + 1$. A *critical graph* is a connected graph G such that G is of class 2 and $G - e$ is of class 1 for each edge $e \in E$. A critical graph of maximum degree Δ is called a Δ-*critical graph*. It is obvious that, for $k \geq 2$, every k-critical graph is 2-connected.

On the edge colorability of plane graphs, Vizing [4] first showed that every plane graph with $\Delta \geq 8$ is of class 1. In the same paper, Vizing gave examples of plane graphs of class 2 for each $\Delta \in \{2, 3, 4, 5\}$ and conjectured that every plane graph with $6 \leq \Delta \leq 7$ is edge Δ-colorable. The case $\Delta = 7$ of the conjecture was confirmed by Zhang [6], and by Sanders and

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Zhao [2], independently. For the only open case $\Delta = 6$ of the conjecture, a number of authors have reported some results. Zhou [7] first showed that every plane graph with $\Delta = 6$ and without k-cycles for some fixed $k \in \{3, 4, 5\}$ is of class 1. Bu and Wang [1] and Wang and Chen [5] extended Zhou's result by showing that every plane graph G with $\Delta = 6$ is of class 1 if it satisfies one of the following conditions:

(1) it is without 6-cycles;
(2) it is without chordal 4-cycles;
(3) it is without chordal 5-cycles.

In this note, we show the following.

Theorem 1. *Every plane graph G with $\Delta = 6$ is of class 1 if no vertex in G is incident with four 3-faces.*

This clearly implies that every plane graph with $\Delta = 6$ and without adjacent triangles (i.e., without chordal 4-cycles) is of class 1, and provides new evidence supporting the planar graph conjecture.

2. Structure of critical graphs

Suppose that Theorem 1 is false. Let $G = (V, E)$ be a counterexample to Theorem 1 with the fewest edges, i.e., G is 6-critical. Below are some known structural lemmas about critical graphs that were frequently used in the earlier papers.

Lemma 1 (*Vizing's Adjacency Lemma [4]*)**.** *Let G be a Δ-critical graph. Then the following hold.*

(1) *Any vertex of G is adjacent to at least two Δ-vertices.*
(2) *If $x \in V$ with $d_k(x) \geq 1$, where $k \neq \Delta$, then $d_\Delta(x) \geq \Delta - k + 1$.*

Lemma 2 below follows directly from Lemma 1.

Lemma 2. *Let G be a 6-critical graph, and let $v \in V$.*

(1) *If $d(v) = 2$, then v is adjacent to two 6-vertices.*
(2) *If $d(v) = 3$, then v has no 4^--neighbors.*
(3) *If $d(v) = 4$, then v has no 3^--neighbors.*
(4) *Let $d(v) = 5$. If $d_3(v) = 1$, then $d_6(v) = 4$; and if $d_4(v) \geq 1$, then $d_6(v) \geq 3$.*
(5) *Let $d(v) = 6$. If $d_2(v) = 1$, then $d_6(v) = 5$; if $d_3(v) \geq 1$, then $d_6(v) \geq 4$; and if $d_4(v) \geq 1$, then $d_6(v) \geq 3$.*

Lemma 3 (*Zhang [6]*)**.** *Let G be a critical graph with maximum degree Δ. If $xy \in E$ and $d(x) + d(y) = \Delta + 2$, then the following hold.*

(1) *Every vertex in $N(x, y) \setminus \{x, y\}$ is a Δ-vertex.*
(2) *Every vertex in $N(N(x, y)) \setminus \{x, y\}$ is of degree at least $\Delta - 1$.*
(3) *If both $d(x), d(y) < \Delta$, then every vertex in $N(N(x, y)) \setminus \{x, y\}$ is a Δ-vertex.*

By Lemmas 2 and 3, we have the following.

Corollary 1. *If $[xyz]$ is a 3-face in G, then $d(x) + d(y) + d(z) \geq 14$.*

Lemma 4 (*Sanders and Zhao [2]*)**.** *No Δ-critical graph has distinct vertices x, y, z such that x is adjacent to y and z, $d(z) < 2\Delta - d(x) - d(y) + 2$, and xz is in at least $d(x) + d(y) - \Delta - 2$ triangles not containing y.*

3. Discharging

Recall that $G = (V, E, F)$ is a counterexample to Theorem 1 with the fewest edges, and is hence a 6-critical graph having all structural properties described in Lemmas 1 to 4 above. To complete the proof of Theorem 1, we shall use these structural properties to derive a contradiction by a discharging procedure carried out in G. The *initial charge function* ch in the discharging procedure is defined as follows: $\mathrm{ch}(x) = d(x) - 4$ for every $x \in V \cup F$.

By the handshaking lemma $\sum_{v \in V} d(v) = 2|E| = \sum_{f \in F} d(f)$ and Euler's formula $|V| - |E| + |F| = 2$, we have

$$\sum_{v \in V \cup F} (d(x) - 4) = -8.$$

Since any discharging procedure preserves the total charge of G, if we can define suitable discharging rules to change the initial charge function ch to the *final charge function* ch′ on $V \cup F$ such that $\mathrm{ch}'(x) \geq 0$ for all $x \in V \cup F$, then $0 \leq \sum_{x \in V \cup F} \mathrm{ch}'(x) = \sum_{x \in V \cup F} \mathrm{ch}(x) = -8$, a contradiction completing the proof of Theorem 1.

Now, let us introduce the following required discharging rules.

R1. *Charge to a* 2-*vertex* v
Every 6-neighbor of v sends 1 to v.

R2. *Charge to a* 3-*vertex* v

R2.1. If v has exactly two 6-neighbors, then it gets $\frac{1}{2}$ from each of its 6-neighbors.

R2.2. If v has three 6-neighbors, then it gets $\frac{1}{3}$ from each of its 6-neighbors.

R3. *Charge to a* 3-*face* f

R3.1. If f is a $(4^-, 6, 6)$-face, then it gets $\frac{1}{2}$ from each of its 6-vertices.

R3.2. If f is a $(4^-, 5, 6)$-face, then it gets $\frac{1}{3}$ from its 5-vertex and $\frac{2}{3}$ from its 6-vertex.

R3.3. If f is a $(4, 4, 6)$-face, then it gets 1 from its 6-vertex.

R3.4. If f is a $(4, 5, 5)$-face, then it gets $\frac{1}{2}$ from each of its 5-vertices.

R3.5. If f is a $(5^+, 5^+, 5^+)$-face, then it gets $\frac{1}{3}$ from each of its 5^+-vertices.

R4. *Charge to a* 5-*vertex* v
If v is incident with a $(4, 5, 5)$-face, then it gets $\frac{1}{18}$ from each of its 6-neighbors.

R5. *Charge to a* 6-*vertex* v
If v is incident with a $(2, 6, 6)$-face, then it gets $\frac{1}{24}$ from each of its 6-neighbors which are not in this $(2, 6, 6)$-face.

The rest of this paper is devoted to checking that $\mathrm{ch}'(x) \geq 0$ for all $x \in V \cup F$.

We first check the final charge for $f \in F$.

If $d(f) \geq 4$, then no charge is discharged to or from f by our rules. Hence $\mathrm{ch}'(f) = \mathrm{ch}(f) \geq 4 - 4 = 0$.

Suppose that $d(f) = 3$ and $f = [xyz]$ with $d(x) \leq d(y) \leq d(z)$. Since G is 2-connected, $d(x) \geq 2$. If $d(x) = 2$, then $d(y) = d(z) = 6$ by Corollary 1. By R3.1, f gets $\frac{1}{2}$ from each of y and z, giving $\mathrm{ch}'(f) = \mathrm{ch}(f) + 2 \times \frac{1}{2} = 3 - 4 + 1 = 0$. Suppose that $d(x) = 3$. By Corollary 1, $5 \leq d(y) \leq 6$ and $d(z) = 6$. If $d(y) = 5$, then $\mathrm{ch}'(f) = \mathrm{ch}(f) + \frac{1}{3} + \frac{2}{3} = 3 - 4 + 1 = 0$ by R3.2; if $d(y) = 6$, then $\mathrm{ch}'(f) = \mathrm{ch}(f) + 2 \times \frac{1}{2} = 3 - 4 + 1 = 0$ by R3.1. Suppose that $d(x) = 4$. By Corollary 1, $4 \leq d(y) \leq 6$. If $d(y) = 4$, then $d(z) = 6$ by Corollary 1, and we have $\mathrm{ch}'(f) = \mathrm{ch}(f) + 1 = 0$ by R3.3. If $d(y) = 5$, then $5 \leq d(z) \leq 6$. We have $\mathrm{ch}'(f) = \mathrm{ch}(f) + 2 \times \frac{1}{2} = 0$ by R3.4 when $d(z) = 5$; and $\mathrm{ch}'(f) = \mathrm{ch}(f) + \frac{1}{3} + \frac{2}{3} = 0$ by R3.2 when $d(z) = 6$. If $d(y) = 6$, then $d(z) = 6$. We have $\mathrm{ch}'(f) = \mathrm{ch}(f) + 2 \times \frac{1}{2} = 0$ by R3.1. Finally, suppose that $d(x) \geq 5$. By R3.5, we have $\mathrm{ch}'(f) = \mathrm{ch}(f) + 3 \times \frac{1}{3} = 0$.

We next analyze the final charge for $v \in V$.

Let $d(v) = 2$. By Lemma 2(1), $d_6(v) = 2$. By R1, $\mathrm{ch}'(v) = \mathrm{ch}(v) + 2 \times 1 = 2 - 4 + 2 = 0$.

Let $d(v) = 3$. By Lemmas 2 and 1(1), every neighbor of v is a 5^+-vertex and $d_6(v) \geq 2$. If $d_6(v) = 2$, then, by R2.1, $\mathrm{ch}'(v) = \mathrm{ch}(v) + 2 \times \frac{1}{2} = 3 - 4 + 1 = 0$. Otherwise, by R2.2, $\mathrm{ch}'(v) = \mathrm{ch}(v) + 3 \times \frac{1}{3} = 3 - 4 + 1 = 0$.

Let $d(v) = 4$. By our rules, no charge is discharged to or from v; hence $\mathrm{ch}'(v) = \mathrm{ch}(v) = 4 - 4 = 0$.

Let $d(v) = 5$. First note that v is incident with at most three 3-faces by our assumption. If no $(4, 5, 5)$-face is incident with v, then v sends at most $\frac{1}{3}$ to each incident 3-face by R3; hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 3 \times \frac{1}{3} = 0$. Suppose that v is incident with at least one $(4, 5, 5)$-face. Observe that v has a 4-neighbor. By Lemma 2, $d_6(v) = 3$. By R4, every 6-neighbor of v sends $\frac{1}{18}$ to v, yielding $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - \frac{1}{2} - 2 \times \frac{1}{3} + 3 \times \frac{1}{18} = 0$.

Finally, let $d(v) = 6$.

We first assume that no 3-face is incident with v. In this case, R3 does not apply. Namely, v only needs to send charge to its non-4-neighbors. If $d_{4^-}(v) = 0$, by R4 and R5, $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - d_5(v) \times \frac{1}{18} - d_6(v) \times \frac{1}{24} \geq 6 - 4 - d_5(v) \times \frac{1}{18} - (6 - d_5(v)) \times \frac{1}{24} \geq \frac{7}{4} - d_5(v) \times \frac{1}{72} \geq \frac{7}{4} - \frac{1}{12} > 0$. Suppose that $d_{4^-}(v) \geq 1$. First note that, by Lemma 3(2), each of the 6-neighbors of v is not incident with any $(2, 6, 6)$-face, namely R5 does not apply, or v sends nothing to its 6-neighbors. If $d_2(v) \geq 1$, then $d_2(v) = 1$ and $d_6(v) = 5$ by Lemma 2; hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 1 = 6 - 4 - 1 = 1 > 0$ by R1. So we may assume that $d_2(v) = 0$. If $d_3(v) \geq 1$, then $d_3(v) \leq 2$ and $d_6(v) \geq 4$ by Lemma 2; hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - \max\{2 \times \frac{1}{2}, \frac{1}{2} + \frac{1}{18}\} \geq 6 - 4 - 1 = 1 > 0$ by R2 and R4. So we may assume that $d_{3^-}(v) = 0$. Now, $d_4(v) \geq 1$. By Lemma 2, $d_6(v) \geq 3$. We have $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 2 \times \frac{1}{18} \geq 6 - 4 - \frac{1}{9} > 0$.

We next assume that v is incident with at least one 3-face. Recall that v is incident with at most three 3-faces. First, suppose that v is incident with at least one $(4, 4, 6)$-face, namely v has at least two adjacent 4-neighbors. By Lemma 3(3), $d_6(v) = 4$. Furthermore, each of these four 6-neighbors of v is not incident with any $(2, 6, 6)$-face by Lemma 3(2); hence R5 does not apply. By R3, $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 1 - 2 \times \frac{1}{2} = 0$. So, from now on, we suppose that v is not incident with any $(4, 4, 6)$-face; hence, in the later, v sends at most $\frac{2}{3}$ to each of the incident 3-faces by R3.

Suppose that v is incident with a $(4^-, 5, 6)$-face. By Lemma 3(2), each of the 6-neighbors of v is not incident with any $(2, 6, 6)$-face; hence v sends nothing to its 6-neighbors. If the $(4^-, 5, 6)$-face is a $(3, 5, 6)$-face, then $d_6(v) = 4$ by Lemma 3; hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - \frac{1}{2} - \frac{2}{3} - \frac{1}{2} - \frac{1}{3} = 0$. If the $(4^-, 5, 6)$-face is a $(4, 5, 6)$-face, then $d_6(v) \geq 3$ by Lemma 2; hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 2 \times \frac{2}{3} - \frac{1}{2} - 2 \times \frac{1}{18} \geq \frac{1}{18} > 0$. From now on, we may assume that v is also not incident with any $(4^-, 5, 6)$-face, and hence it sends at most $\frac{1}{2}$ to each of the incident 3-faces by R3.

Suppose that v is incident with a $(2, 6, 6)$-face. By Lemma 2, $d_6(v) = 5$. By R5, each of the 6-neighbors of v not on the $(2, 6, 6)$-face sends $\frac{1}{24}$ to v. Hence, $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 1 - \frac{1}{2} - 2 \times \frac{1}{3} + 4 \times \frac{1}{24} = 0$. From now on, v is also not incident with any $(2, 6, 6)$-face.

Suppose that v is incident with a (3, 6, 6)-face, denoted $[xyv]$ with $d(x) = 3$. By Lemma 2, $d_6(v) \geq 4$. Let y_1, y_2, y_3 be three 6-neighbors of v other than y, and let u be the remaining neighbor of v. By Lemma 2(5), u is a 3^+-vertex. If $d(u) = 3$ or $d(u) = 4$, uv does not belong to any 3-face by Lemma 4, and the 3-neighbor(s) of v is (or are) adjacent to three 6-vertices by Lemma 3; hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 2 \times \frac{1}{3} - 2 \times \frac{1}{2} - \frac{1}{3} = 0$ by R2 and R3. If $d(u) \geq 5$, then $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - \frac{1}{2} - 2 \times \frac{1}{2} - \frac{1}{3} - \frac{1}{18} = \frac{1}{9} > 0$ by R2–R4. From now on, v is also not incident with any (3, 6, 6)-face.

Suppose that v is incident with a (4, 6, 6)-face. By Lemma 2(5), $d_6(v) \geq 3$, and the remaining two neighbors of v are 3^+-vertices. Furthermore, $d_3(v) \leq 1$ by Lemma 2(5). By Lemma 3(2), the 6-neighbors of v are not incident with any (2, 6, 6)-face. If $d_3(v) = 0$, then $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 3 \times \frac{1}{2} - 2 \times \frac{1}{18} = \frac{7}{18} > 0$ by R3 and R4. If $d_3(v) = 1$, then v has four 6-neighbors; hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - \frac{1}{2} - 3 \times \frac{1}{2} = 0$ by R2 and R3.

Now every 3-face incident with v is a $(5^+, 5^+, 6)$-face. If $d_{3^-}(v) = 0$, then $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 3 \times \frac{1}{3} - d_5(v) \times \frac{1}{18} - d_6(v) \times \frac{1}{24} \geq 6 - 4 - 1 - d_5(v) \times \frac{1}{18} - (6 - d_5(v)) \times \frac{1}{24} \geq \frac{3}{4} - d_5(v) \times \frac{1}{72} \geq \frac{3}{4} - \frac{1}{12} > 0$ by R4 and R5. Suppose that $d_{3^-}(v) \geq 1$. By Lemma 3, the 6-neighbors of v are not incident with any (2, 6, 6)-face; hence R5 does not apply. If $d_2(v) \geq 1$, then $d_2(v) = 1$ and $d_6(v) = 5$ by Lemma 2(5); hence $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 1 - 3 \times \frac{1}{3} = 0$ by R1 and R3. If $d_2(v) = 0$, then $d_3(v) \geq 1$ and $d_6(v) \geq 4$ by Lemma 2; hence $d_3(v) \leq 2$ and $\mathrm{ch}'(v) \geq \mathrm{ch}(v) - 3 \times \frac{1}{3} - \max\{2 \times \frac{1}{2}, \frac{1}{2} + \frac{1}{18}\} \geq 0$ by R2–R4.

The proof of Theorem 1 is completed. □

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